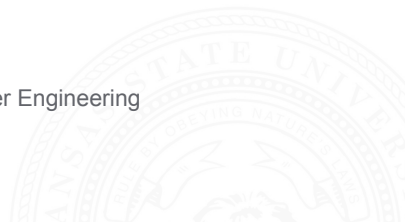


Power Flow II

Construction of the bus admittance matrix

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Lecture outline

- 1 Write the bus equations, form their matrix, then construct Y-bus for a general network.
- 2 Extend the matrix for shunts; transformer tap corrections are optional.
- 3 Calculate Y-bus and bus power injections, then update the matrix for added equipment.

PART 1

Building Y-bus from the network

Write the known three-bus equations in matrix form. Interpret their coefficients, then extend the construction to a general network.

Why use matrix form?

Branch current relations and KCL already give an equation at each bus. For example,

$$I_1 = y_{12}(V_1 - V_2) + y_{13}(V_1 - V_3).$$

A large network

Each bus has different connections. Managing a separate expression for every bus becomes cumbersome as the network grows.

Computer implementation

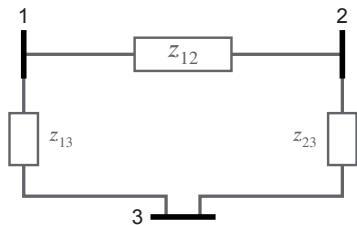
Store voltage coefficients in a matrix. Software assembles them from branch data, then evaluates all bus currents with one matrix operation.

$$\mathbf{I} = \mathbf{Y}_{\text{bus}} \mathbf{V}$$

$\mathbf{I}, \mathbf{V}$: vectors of bus currents and voltages. $\mathbf{Y}_{\text{bus}}$: bus admittance matrix.

Current equations at the three buses

Series equivalent on the common base



$$y_{ik} = 1/z_{ik}$$

At each bus, net current injection equals the sum of currents into its connected branches.

$$I_1 = y_{12}(V_1 - V_2) + y_{13}(V_1 - V_3),$$

$$I_2 = y_{12}(V_2 - V_1) + y_{23}(V_2 - V_3),$$

$$I_3 = y_{13}(V_3 - V_1) + y_{23}(V_3 - V_2).$$

Series model: unity taps, no charging or bus shunts.

Collect the voltage coefficients

Expand each equation and collect terms in the order V_1, V_2, V_3 .

For bus 1,

$$\begin{aligned} I_1 &= y_{12}V_1 - y_{12}V_2 + y_{13}V_1 - y_{13}V_3 \\ &= (y_{12} + y_{13})V_1 - y_{12}V_2 - y_{13}V_3. \end{aligned}$$

Applying the same step at buses 2 and 3 gives

$$\begin{aligned} I_2 &= -y_{12}V_1 + (y_{12} + y_{23})V_2 - y_{23}V_3, \\ I_3 &= -y_{13}V_1 - y_{23}V_2 + (y_{13} + y_{23})V_3. \end{aligned}$$

Arrange the equations in matrix form

Each current equation gives one row. The coefficients of V_1 , V_2 , and V_3 fill columns 1, 2, and 3.

$$\begin{bmatrix} I_1 \\ I_2 \\ I_3 \end{bmatrix} = \underbrace{\begin{bmatrix} y_{12} + y_{13} & -y_{12} & -y_{13} \\ -y_{12} & y_{12} + y_{23} & -y_{23} \\ -y_{13} & -y_{23} & y_{13} + y_{23} \end{bmatrix}}_{\mathbf{Y}_{\text{bus}}^{\text{ser}}} \begin{bmatrix} V_1 \\ V_2 \\ V_3 \end{bmatrix}.$$

For example, row 2 reproduces the bus 2 equation:

$$I_2 = -y_{12}V_1 + (y_{12} + y_{23})V_2 - y_{23}V_3.$$

Self and mutual admittances

The coefficients in the three-bus equations form the complete series matrix:

$$\mathbf{Y}_{\text{bus}}^{\text{ser}} = \begin{bmatrix} y_{12} + y_{13} & -y_{12} & -y_{13} \\ -y_{12} & y_{12} + y_{23} & -y_{23} \\ -y_{13} & -y_{23} & y_{13} + y_{23} \end{bmatrix}.$$

Diagonal: self admittance

$Y_{11} = y_{12} + y_{13}$ multiplies V_1 in the bus 1 equation. Both branches at bus 1 contribute.

Off-diagonal: mutual admittance

$Y_{12} = -y_{12}$ multiplies V_2 in the bus 1 equation. It comes from the connection between buses 1 and 2.

The matrix for a general network

For n buses, write all n current equations in the same form:

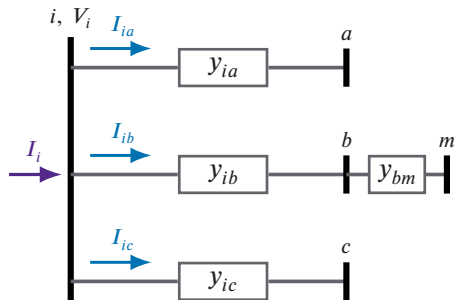
$$\underbrace{\begin{bmatrix} I_1 \\ I_2 \\ \vdots \\ I_n \end{bmatrix}}_{\mathbf{I}} = \underbrace{\begin{bmatrix} Y_{11} & Y_{12} & \cdots & Y_{1n} \\ Y_{21} & Y_{22} & \cdots & Y_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ Y_{n1} & Y_{n2} & \cdots & Y_{nn} \end{bmatrix}}_{\mathbf{Y}_{\text{bus}}} \underbrace{\begin{bmatrix} V_1 \\ V_2 \\ \vdots \\ V_n \end{bmatrix}}_{\mathbf{V}}.$$

Y_{ik} is the coefficient of V_k in the equation for I_i .

$$I_i = Y_{ii}V_i + \sum_{k \neq i} Y_{ik}V_k$$

The same self and mutual admittance rules determine every row.

Diagonal entries in a general network



Other network connections omitted

Add the series admittances connected to bus i :

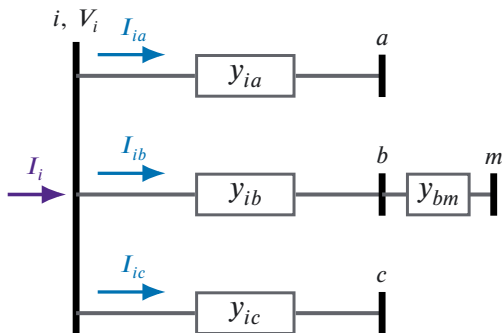
$$Y_{ii} = \sum_{k \in \mathcal{N}_i} y_{ik}$$

$\mathcal{N}_i$: buses directly connected to i .

y_{ik} : total series admittance linking buses i and k .

For the network shown, $Y_{ii} = y_{ia} + y_{ib} + y_{ic}$.

Off-diagonal entries in a general network



Other network connections omitted

Mutual admittance between i and k

Use the negative admittance of the direct connection:

$$Y_{ik} = \begin{cases} -y_{ik}, & k \in \mathcal{N}_i, \\ 0, & k \notin \mathcal{N}_i, k \neq i. \end{cases}$$

For the connections shown,

$$Y_{ib} = -y_{ib}, \quad Y_{im} = 0.$$

Bus m connects through b , so there is no direct i – m coefficient.

Construct Y-bus: diagonal, then off-diagonal

Number the n buses and use the same order for rows and columns.

1. Write the diagonal entries

For each bus, add the admittances of all branches connected to it.

$$Y_{ii} = \sum_{k \in \mathcal{N}_i} y_{ik}$$

Fill $(1, 1), (2, 2), \dots, (n, n)$.

2. Write the off-diagonal entries

For each pair of different buses, use the negative connecting admittance, or zero if they are not directly connected.

$$Y_{ik} = \begin{cases} -y_{ik}, & k \in \mathcal{N}_i, \\ 0, & k \notin \mathcal{N}_i, k \neq i. \end{cases}$$

Calculate the bus current vector

Given the bus voltages, evaluate all current injections with the assembled Y-bus.

Compact form

$$\mathbf{I} = \mathbf{Y}_{\text{bus}} \mathbf{V}$$

$\mathbf{I}$ and $\mathbf{V}$ each contain
one entry per bus.

Expanded matrix form

$$\begin{bmatrix} I_1 \\ I_2 \\ \vdots \\ I_n \end{bmatrix} = \begin{bmatrix} Y_{11} & Y_{12} & \cdots & Y_{1n} \\ Y_{21} & Y_{22} & \cdots & Y_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ Y_{n1} & Y_{n2} & \cdots & Y_{nn} \end{bmatrix} \begin{bmatrix} V_1 \\ V_2 \\ \vdots \\ V_n \end{bmatrix}$$

Each matrix row evaluates one bus current:

$$I_i = Y_{i1}V_1 + Y_{i2}V_2 + \cdots + Y_{in}V_n = \sum_{k=1}^n Y_{ik}V_k.$$

Calculate the bus power vector

Collect the individual bus powers $S_i = V_i I_i^*$ into one column vector.

Compact form

$$\mathbf{S} = \text{diag}(\mathbf{V})\mathbf{I}^*$$

$$\mathbf{S} = \mathbf{P} + j\mathbf{Q}$$

Expanded matrix form

$$\begin{bmatrix} S_1 \\ S_2 \\ \vdots \\ S_n \end{bmatrix} = \begin{bmatrix} V_1 & 0 & \cdots & 0 \\ 0 & V_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & V_n \end{bmatrix} \begin{bmatrix} I_1^* \\ I_2^* \\ \vdots \\ I_n^* \end{bmatrix}$$

Only the diagonal coefficient in row i is nonzero, so $S_i = V_i I_i^*$ without summing powers across buses.

$$\mathbf{S} = \text{diag}(\mathbf{V})(\mathbf{Y}_{\text{bus}}\mathbf{V})^*$$

* conjugates each vector element. The real and imaginary parts of $\mathbf{S}$ give $\mathbf{P}$ and $\mathbf{Q}$.

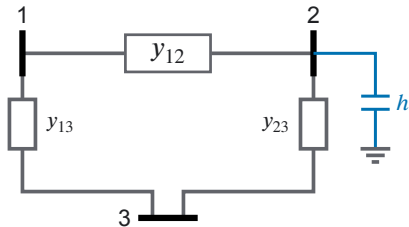
PART 2

Extending an existing Y-bus

Use the three-bus network to introduce shunt and transformer modifications, then apply each update to a general network.

Add one shunt at bus 2

Add a capacitive shunt with admittance h at bus 2.



The additional current is

$$\Delta I_2 = hV_2.$$

Only the coefficient of V_2 in the bus 2 equation changes:

$$Y_{22}^{\text{new}} = Y_{22}^{\text{ser}} + h$$

$$\mathbf{Y}_{\text{bus}}^{\text{new}} = \mathbf{Y}_{\text{bus}}^{\text{ser}} + \begin{bmatrix} 0 & 0 & 0 \\ 0 & h & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

Add shunts to a general network

Shunt admittances are added only to the diagonal entries of Y-bus.

1. Identify the shunts at each bus

Include bus shunts and the charging admittance at each connected line end.

2. Add their admittances at that bus

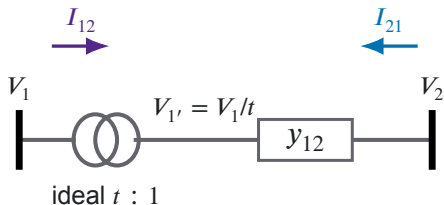
Let h_i be the total added shunt admittance at bus i . If there is no shunt, $h_i = 0$.

$$Y_{11}^{\text{new}} = Y_{11}^{\text{old}} + h_1, \quad Y_{22}^{\text{new}} = Y_{22}^{\text{old}} + h_2, \quad \dots, \quad Y_{nn}^{\text{new}} = Y_{nn}^{\text{old}} + h_n.$$

$$\mathbf{Y}_{\text{bus}}^{\text{new}} = \mathbf{Y}_{\text{bus}}^{\text{old}} + \begin{bmatrix} h_1 & 0 & \dots & 0 \\ 0 & h_2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & h_n \end{bmatrix}.$$

Transformer tap: matrix coefficients (optional)

Use a real tap ratio $t > 0$ on the bus 1 side, with no phase shift. Neglect charging and excitation.



Replace the branch contributions

Entry	Original	With tap
Y_{11}	y_{12}	y_{12}/t^2
Y_{22}	y_{12}	y_{12}
Y_{12}	$-y_{12}$	$-y_{12}/t$
Y_{21}	$-y_{12}$	$-y_{12}/t$

Derivation is provided in the backup slides.

Keep all contributions from other branches and bus shunts.

Extended Y-bus (optional)

The same three-bus network now has shunt h at bus 2 and real tap t on the bus 1 side of branch 1-2:

$$\mathbf{Y}_{\text{bus}} = \begin{bmatrix} \frac{y_{12}}{t^2} + y_{13} & -\frac{y_{12}}{t} & -y_{13} \\ -\frac{y_{12}}{t} & y_{12} + y_{23} + h & -y_{23} \\ -y_{13} & -y_{23} & y_{13} + y_{23} \end{bmatrix}.$$

Setting $t = 1$ and $h = 0$ recovers the original series network matrix.

Updating an existing Y-bus

1 Start with the series network matrix

Keep the bus order and the diagonal and off-diagonal entries already constructed in Part 1.

2 Add shunt admittances

At each bus, add line charging and bus shunt admittances to its diagonal entry. The off-diagonal entries stay unchanged.

3 Replace transformer branch contributions (optional)

Update the diagonal entries first, then the off-diagonal entries. Replace only this branch's coefficients and retain all other device contributions.

$$\mathbf{Y}_{\text{bus}}^{\text{new}} = \mathbf{Y}_{\text{bus}}^{\text{ser}} + \Delta \mathbf{Y}_{\text{bus}}^{\text{sh}} + \Delta \mathbf{Y}_{\text{bus}}^{\text{tap}}$$

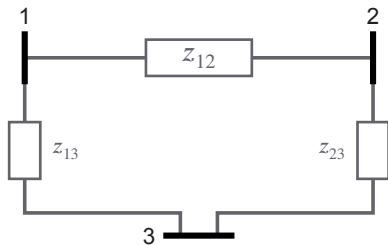
PART 3

Numerical example and extension

Assemble Y-bus, calculate bus currents and powers, then update the matrix for shunts and a transformer tap.

Example 1: original network

Series equivalent on the common base



Branch	z_{ik} (p.u.)
1–2	$0.018 + j0.060$
1–3	$0.070 + j0.210$
2–3	$0.045 + j0.135$

All impedances use a common 100 MVA base.

Use bus order (1, 2, 3) for the matrix rows and columns.

Find the complete 3×3 Y-bus with unity taps, no charging, and no bus shunts.

Example 1: branch admittances

For branch 1–2,

$$y_{12} = \frac{1}{0.018 + j0.060} = \frac{0.018 - j0.060}{0.018^2 + 0.060^2} = 4.587156 - j15.290520.$$

Branch	Admittance (p.u.)
1–2	4.587156 – j15.290520
1–3	1.428571 – j4.285714
2–3	2.222222 – j6.666667

Retain the original reciprocals for calculations. Displayed values are rounded.

Example 1: diagonal entries

At each bus, add the admittances of its connected branches.

$$Y_{11} = y_{12} + y_{13} = 6.015727 - j19.576234,$$

$$Y_{22} = y_{12} + y_{23} = 6.809378 - j21.957187,$$

$$Y_{33} = y_{13} + y_{23} = 3.650794 - j10.952381.$$

Each branch contributes its admittance to the diagonal at both endpoint buses.

Example 1: off-diagonal entries

Next, fill the entries linking different buses. Each branch contributes to the two corresponding off-diagonal positions.

$$Y_{12} = Y_{21} = -y_{12} = -4.587156 + j15.290520,$$

$$Y_{13} = Y_{31} = -y_{13} = -1.428571 + j4.285714,$$

$$Y_{23} = Y_{32} = -y_{23} = -2.222222 + j6.666667.$$

Each mutual entry is the negative admittance of its connecting branch.

Example 1: complete Y-bus

With bus order (1, 2, 3), the original series network gives

$$\mathbf{Y}_{\text{bus}}^{\text{ser}} = \begin{bmatrix} 6.015727 - j19.576234 & -4.587156 + j15.290520 & -1.428571 + j4.285714 \\ -4.587156 + j15.290520 & 6.809378 - j21.957187 & -2.222222 + j6.666667 \\ -1.428571 + j4.285714 & -2.222222 + j6.666667 & 3.650794 - j10.952381 \end{bmatrix}.$$

$$\begin{bmatrix} I_1 \\ I_2 \\ I_3 \end{bmatrix} = \mathbf{Y}_{\text{bus}}^{\text{ser}} \begin{bmatrix} V_1 \\ V_2 \\ V_3 \end{bmatrix}$$

All entries are in per unit. Next, use this matrix to calculate bus current and power injections.

Example 1: calculate the current vector

Use the completed $\mathbf{Y}_{bus}^{ser}$ with the trial voltage vector (p.u.)

$$\mathbf{V} = \begin{bmatrix} 1.02 \\ 1.00 \\ 0.98 - j0.02 \end{bmatrix}.$$

One matrix multiplication gives all bus current injections:

$$\boxed{\mathbf{I} = \mathbf{Y}_{bus}^{ser} \mathbf{V}} \Rightarrow \mathbf{I} = \begin{bmatrix} 0.234600 - j0.448668 \\ 0.086035 + j0.216922 \\ -0.320635 + j0.231746 \end{bmatrix} \text{ p.u.}$$

For a fixed network, the computer reuses the same Y-bus whenever the trial voltage vector changes.

Example 1: power injections in matrix form

Conjugate the current vector element by element, then multiply by the voltage diagonal:

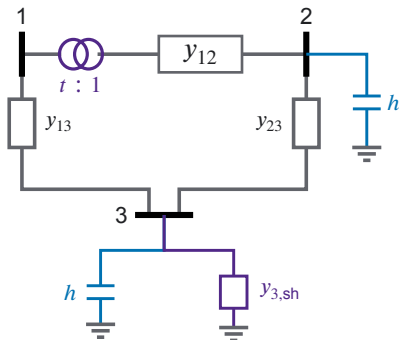
$$\mathbf{S} = \mathbf{P} + j\mathbf{Q} = \text{diag}(\mathbf{V})\mathbf{I}^* = \text{diag}(\mathbf{V})(\mathbf{Y}_{\text{bus}}^{\text{ser}}\mathbf{V})^*$$

$$\mathbf{S} = \begin{bmatrix} 1.02 & 0 & 0 \\ 0 & 1.00 & 0 \\ 0 & 0 & 0.98 - j0.02 \end{bmatrix} \begin{bmatrix} 0.234600 + j0.448668 \\ 0.086035 - j0.216922 \\ -0.320635 - j0.231746 \end{bmatrix} = \begin{bmatrix} 0.239292 + j0.457641 \\ 0.086035 - j0.216922 \\ -0.318857 - j0.220698 \end{bmatrix} \text{ p.u.}$$

$$\mathbf{P} = \text{Re}(\mathbf{S}), \quad \mathbf{Q} = \text{Im}(\mathbf{S})$$

A power flow solver repeats these vector calculations while adjusting voltages to satisfy the required power equations.

Example 2: equipment additions



Transformer on branch 1–2 (optional)

$t = 1.05$ on the bus 1 side. Retain the given series impedance, with no charging or excitation.

Line 2–3

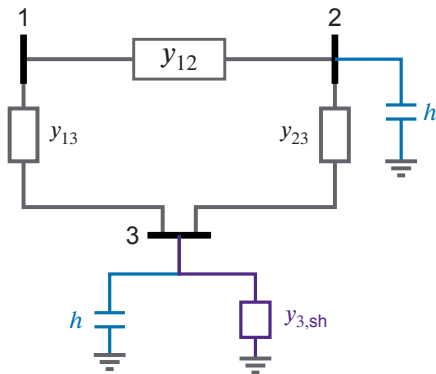
Total charging $b_{c,23} = 0.04$ p.u. Each end contributes $h = j0.02$ p.u.

Bus 3 shunt

$y_{3,sh} = j0.12$ p.u.

Keep the branch impedances from Example 1. Update its Y-bus for these equipment additions.

Example 2: charging and bus shunt



Line 2–3 adds $j0.02$ to both Y_{22} and Y_{33} .
The bus 3 shunt adds another $j0.12$ to Y_{33} .

$$\Delta \mathbf{Y}_{\text{bus}}^{c+\text{sh}} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & j0.02 & 0 \\ 0 & 0 & j0.14 \end{bmatrix}$$

$$Y_{22}^{\text{new}} = Y_{22}^{\text{ser}} + j0.02 = 6.809378 - j21.937187,$$

$$Y_{33}^{\text{new}} = Y_{33}^{\text{ser}} + j0.14 = 3.650794 - j10.812381.$$

Example 2: tap diagonal entries (optional)

Branch 1–2 has $t = 1.05$ on the bus 1 side. First update its self contributions.

Bus 1

Replace y_{12} with $y_{12}/1.05^2$:

$$Y_{11}^{\text{new}} = Y_{11}^{\text{ser}} - y_{12} + \frac{y_{12}}{1.05^2} = 5.589257 - j18.154667.$$

Bus 2

The transformer keeps the self contribution y_{12} . Retain the charging already added:

$$Y_{22}^{\text{new}} = Y_{22}^{\text{ser}} + j0.02 = 6.809378 - j21.937187.$$

The previously updated Y_{33} is unchanged by this transformer.

Example 2: tap mutual entries (optional)

Next, replace $-y_{12}$ in positions (1, 2) and (2, 1).

With $t = 1.05$, both mutual contributions are $-y_{12}/t$:

$$Y_{12}^{\text{new}} = -\frac{y_{12}}{1.05} = -4.368720 + j14.562400,$$

$$Y_{21}^{\text{new}} = -\frac{y_{12}}{1.05} = -4.368720 + j14.562400.$$

The mutual entries for branches 1–3 and 2–3 remain unchanged.

Example 2: updated Y-bus (optional)

Combine the charging, bus shunt, and tap modifications:

$$\mathbf{Y}_{\text{bus}}^{\text{new}} = \begin{bmatrix} 5.589257 - j18.154667 & -4.368720 + j14.562400 & -1.428571 + j4.285714 \\ -4.368720 + j14.562400 & 6.809378 - j21.937187 & -2.222222 + j6.666667 \\ -1.428571 + j4.285714 & -2.222222 + j6.666667 & 3.650794 - j10.812381 \end{bmatrix}.$$

Purple entries changed from Example 1. The four mutual entries for branches 1–3 and 2–3 remain unchanged.

The diagonal additions represent charging and the bus shunt. The tap changes the branch 1–2 self contribution at bus 1 and its two mutual contributions.

Summary

- 1 Identify self and mutual admittances in the bus current equations.
- 2 Construct diagonal and off-diagonal entries from network connections and device data.
- 3 Add shunt admittances to the diagonal; apply transformer tap corrections as an optional extension.

$$\mathbf{I} = \mathbf{Y}_{\text{bus}} \mathbf{V}$$

Next, use the assembled matrix in $S_i = V_i I_i^*$ to obtain bus power equations and apply operating specifications.

Complex transformer tap derivation

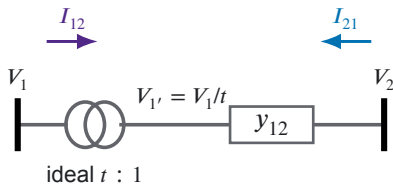
Complex tap ratios and phase shift, port current relations, and diagonal and off-diagonal updates for a general network.



Backup: complex tap and port relations

Allow a complex tap $t = \tau e^{j\theta_t}$ on the bus 1 side, including phase shift θ_t . Neglect charging and excitation.

Internal and external currents



Ideal tap power is conserved:

$$V_1 I_{12}^* = V_{1'} I_{1'2}^*.$$

| The internal voltage V_1/t adds no network bus.

$$I_{1'2} = y_{12}(V_1/t - V_2),$$

$$I_{12} = I_{1'2}/t^*,$$

$$I_{21} = y_{12}(V_2 - V_1/t).$$

First identify each bus's own voltage coefficient. Then identify the coefficient of the other bus voltage.

Backup: diagonal coefficients

The diagonal entry uses the coefficient of a bus's own voltage in its current equation.

Bus 1: the V_1 term in I_{12}

Voltage is divided by t and current by t^* :

$$\frac{y_{12}}{t^*} \frac{V_1}{t} = \frac{y_{12}}{|t|^2} V_1.$$

Replace the old branch contribution y_{12} with $y_{12}/|t|^2$.

Including the other branches and the bus 2 shunt,

$$Y_{11} = \frac{y_{12}}{|t|^2} + y_{13},$$

Bus 2: the V_2 term in I_{21}

The coefficient of V_2 is unchanged:

$$y_{12} V_2.$$

Keep the branch contribution y_{12} at bus 2.

$$Y_{22} = y_{12} + y_{23} + h.$$

Backup: off-diagonal coefficients

Now read the coefficient of the other bus voltage in each current equation.

Bus 1: the V_2 term in I_{12}

Current is divided by t^* :

$$-\frac{1}{t^*} y_{12} V_2 = -\frac{y_{12}}{t^*} V_2.$$

$$Y_{12} = -\frac{y_{12}}{t^*}$$

Bus 2: the V_1 term in I_{21}

Voltage is divided by t :

$$-y_{12} \frac{V_1}{t} = -\frac{y_{12}}{t} V_1.$$

$$Y_{21} = -\frac{y_{12}}{t}$$

Replace the original $-y_{12}$ in these two positions. All other off-diagonal entries stay unchanged.

Backup: general diagonal update

For branch $i-k$, let $y = 1/z_{ik}$ and place tap t on side i .

At the tap-side bus i

Remove the original branch contribution y and insert $y/|t|^2$:

$$Y_{ii}^{\text{new}} = Y_{ii}^{\text{old}} - y + \frac{y}{|t|^2}.$$

At the other bus k

The branch self contribution remains y :

$$Y_{kk}^{\text{new}} = Y_{kk}^{\text{old}} - y + y = Y_{kk}^{\text{old}}.$$

Other branch contributions and bus shunts remain in these diagonal entries.

Backup: general off-diagonal update

For the same branch $i-k$, replace its mutual contributions in the two bus current equations.

Row i , column k

Replace $-y$ with $-y/t^*$:

$$Y_{ik}^{\text{new}} = Y_{ik}^{\text{old}} - (-y) - \frac{y}{t^*} = Y_{ik}^{\text{old}} + y \left(1 - \frac{1}{t^*}\right).$$

Row k , column i

Replace $-y$ with $-y/t$:

$$Y_{ki}^{\text{new}} = Y_{ki}^{\text{old}} - (-y) - \frac{y}{t} = Y_{ki}^{\text{old}} + y \left(1 - \frac{1}{t}\right).$$

All other off-diagonal entries remain unchanged.