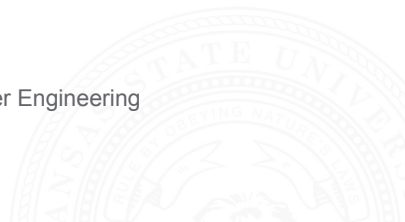


Power Flow III

Power equations, bus types, and a solved case

Buxin She

Mike Wieggers Department of Electrical and Computer Engineering



Lecture outline

- 1 Power equations: express real and reactive power in terms of bus voltages.
- 2 Bus types: count quantities and equations, then identify specified and unknown variables.
- 3 Inductive lines: interpret power transfer and its power-angle maximum.
- 4 Worked example: identify knowns and unknowns, write equations, and interpret a solution.

PART 1

Power equations

Start from one row of the network power relation. Retain the full bus summation, derive each bus P/Q equation, then identify the operating inputs and quantities to be calculated.

Current and power relations

Compact vector form

$$\mathbf{I} = \mathbf{Y}_{\text{bus}} \mathbf{V}$$

$$\mathbf{S} = \text{diag}(\mathbf{V}) \mathbf{I}^*$$

Expanded matrix form

$$\begin{bmatrix} I_1 \\ \vdots \\ I_i \\ \vdots \\ I_n \end{bmatrix} = \begin{bmatrix} Y_{11} & \cdots & Y_{1i} & \cdots & Y_{1n} \\ \vdots & & \vdots & & \vdots \\ Y_{i1} & \cdots & Y_{ii} & \cdots & Y_{in} \\ \vdots & & \vdots & & \vdots \\ Y_{n1} & \cdots & Y_{ni} & \cdots & Y_{nn} \end{bmatrix} \begin{bmatrix} V_1 \\ \vdots \\ V_i \\ \vdots \\ V_n \end{bmatrix}$$

Row i

$$I_i = \sum_{k=1}^n Y_{ik} V_k$$

$$S_i = P_i + jQ_i = V_i I_i^* = V_i \sum_{k=1}^n Y_{ik}^* V_k^*$$

The general bus power equation

Begin with the same equation for bus i in the n -bus network:

$$S_i = P_i + jQ_i = V_i \sum_{k=1}^n Y_{ik}^* V_k^*.$$

Distribute V_i across the sum and retain the self term $k = i$:

$$\begin{aligned} S_i &= \sum_{k=1}^n Y_{ik}^* V_i V_k^* \\ &= |V_i|^2 Y_{ii}^* + \sum_{\substack{k=1 \\ k \neq i}}^n Y_{ik}^* V_i V_k^*. \end{aligned}$$

Y_{ii} is the full self admittance; all Y_{ik} are entries of the assembled Y-bus.

Polar form and conjugation

Write the Y-bus entries and the voltage factors explicitly:

$$Y_{ik} = G_{ik} + jB_{ik}, \quad Y_{ik}^* = G_{ik} - jB_{ik},$$

$$V_i = |V_i|e^{j\theta_i}, \quad V_k^* = |V_k|e^{-j\theta_k}.$$

$$V_i V_k^* = |V_i||V_k|e^{j\theta_{ik}}, \quad \theta_{ik} = \theta_i - \theta_k.$$

Substitute these factors into the same full sum:

$$\begin{aligned} S_i &= \sum_{k=1}^n (G_{ik} - jB_{ik}) |V_i||V_k|e^{j\theta_{ik}} \\ &= |V_i| \sum_{k=1}^n |V_k|(G_{ik} - jB_{ik})e^{j\theta_{ik}}. \end{aligned}$$

Expanding each term

Use $e^{j\theta_{ik}} = \cos \theta_{ik} + j \sin \theta_{ik}$ in each summand:

$$S_i = |V_i| \sum_{k=1}^n |V_k| (G_{ik} - jB_{ik}) (\cos \theta_{ik} + j \sin \theta_{ik}).$$

Multiply the two complex factors for each k :

$$\begin{aligned} & (G_{ik} - jB_{ik}) (\cos \theta_{ik} + j \sin \theta_{ik}) \\ &= G_{ik} \cos \theta_{ik} + jG_{ik} \sin \theta_{ik} - jB_{ik} \cos \theta_{ik} + B_{ik} \sin \theta_{ik} \\ &= (G_{ik} \cos \theta_{ik} + B_{ik} \sin \theta_{ik}) + j(G_{ik} \sin \theta_{ik} - B_{ik} \cos \theta_{ik}). \end{aligned}$$

The $B_{ik} \sin \theta_{ik}$ term is positive because $(-j)(j) = +1$.

Separating real and reactive power

Insert the expanded product and sum the real and imaginary terms separately:

$$S_i = |V_i| \sum_{k=1}^n |V_k| [G_{ik} \cos \theta_{ik} + B_{ik} \sin \theta_{ik}] \\ + j|V_i| \sum_{k=1}^n |V_k| [G_{ik} \sin \theta_{ik} - B_{ik} \cos \theta_{ik}].$$

Compare with the original definition:

$$S_i = P_i + jQ_i, \quad P_i = \operatorname{Re}(S_i), \quad Q_i = \operatorname{Im}(S_i).$$

The first sum gives P_i ; the coefficient of j gives Q_i .

Both sums still run over every $k = 1, \dots, n$, including $k = i$.

Bus injection equations

These two equations at each bus form the power flow model:

$$P_i = |V_i| \sum_{k=1}^n |V_k| [G_{ik} \cos \theta_{ik} + B_{ik} \sin \theta_{ik}]$$

$$Q_i = |V_i| \sum_{k=1}^n |V_k| [G_{ik} \sin \theta_{ik} - B_{ik} \cos \theta_{ik}]$$

Bus variables

$P_i, Q_i, |V_i|, \theta_i$

$\theta_{ik} = \theta_i - \theta_k$; P_i, Q_i are net injections: generation minus local load.

Network parameters

G_{ik}, B_{ik} : fixed Y-bus coefficients

Specified operating data

The operating model supplies powers and voltage targets.

Operating input	Specified quantity
Load demands	$P_{L,i}^{\text{spec}}, Q_{L,i}^{\text{spec}}$
Generator real power schedule	$P_{G,i}^{\text{spec}}$ for scheduled units, $i \neq r$
Regulated voltage magnitude	$ V_i ^{\text{spec}}$
Reference angle	$\theta_r^{\text{spec}} = 0$

r : reference and balancing source. Superscript spec denotes a specified input.

Determined by the network

Use the power equations to find voltages consistent with the specified operating data.

Network quantity	Symbol or relation
Voltage angles except the reference	$\theta_i, \quad i \neq r$
Voltage magnitudes without a target	$ V_i $
Regulated generator output	$Q_{G,i}$
Balancing generator output	$P_{G,r}, \quad Q_{G,r}$
Current injections	$I_i = \sum_{k=1}^n Y_{ik} V_k$

| Solve the unknown voltages, then calculate currents and generator outputs.

PART 2

Bus types

The operating data specify some powers and voltages. Count the bus quantities and equations, identify the remaining unknowns, and name the choices of specified quantities.

Quantities and equations

For a fixed network with n buses, each bus has four real quantities: $|V_i|, \theta_i, P_i, Q_i$.

Item	Network count
Voltage magnitudes and angles	$2n$
Net real and reactive injections	$2n$
Total bus quantities	$4n$
P and Q relations, one pair per bus	$2n$

Currents and branch powers are already eliminated from these bus equations.

| Choose $2n$ operating specifications to leave $2n$ quantities to be determined.

Specified and unknown quantities

The operating model selects two of the four quantities at each bus:

Operating role	Specified	To be determined
Fixed net power	P_i, Q_i	$ V_i , \theta_i$
Regulated generator	$P_i, V_i $	θ_i, Q_i
Reference and balancing source	$ V_r , \theta_r$	P_r, Q_r

Use one reference source in a connected network. The $2n$ specifications leave $2n$ unknown quantities linked by the $2n$ power equations.

Specified powers are net injections: generation minus local load.

| The specified pair determines the bus type.

Bus types and equation selection

- **PQ bus:** specify P and Q .
Solve $|V|$ and θ using both the P and Q balance equations.
- **PV bus:** specify P and regulate $|V|$.
Solve θ using the P balance equation; calculate Q afterward.
- **Slack bus:** specify $|V|$ and the reference angle θ .
No voltage unknowns at this bus; calculate P and Q from the solved network.

Bus types select the specified quantities and voltage equations. Every bus still obeys both power relations.

The power flow problem

For a fixed n -bus network, the bus variables are $|V_i|, \theta_i, P_i, Q_i$.

$$\underbrace{4n}_{\text{bus variables}} - \underbrace{2n}_{\text{specified}} = \underbrace{2n}_{\text{unknown}}$$

There are $2n$ power flow equations: one P and one Q relation per bus.

- **Solve the power flow equations**

Use Y-bus and the operating data to solve for the unknown bus variables.

- **Calculate dependent variables**

Use the solved bus state to calculate branch currents, P/Q flows, and losses.

Use one reference/slack bus for each connected network.

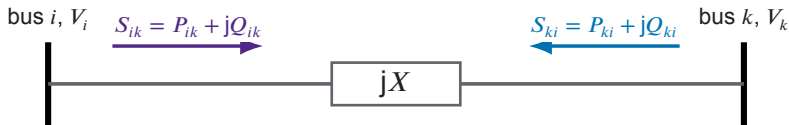
PART 3

Power transfer on an inductive line

Use a transmission line model to see how neglecting resistance simplifies power transfer, then relate real power to angle and reactive power to voltage magnitude.

The inductive line model

For high-voltage transmission lines, resistance is often small compared with reactance: $R \ll X$.



Positive terminal powers enter the line

- Set $R = 0$ to study power transfer through $z_{ik} = jX$, with $X > 0$.
- Use a series model; neglect line charging and shunt elements.

$$y_{ik} = \frac{1}{jX} = -\frac{j}{X} \implies g_{ik} = 0, \quad b_{ik} = -\frac{1}{X}.$$

Real and reactive line power

For a series line, with $\theta_{ik} = \theta_i - \theta_k$:

$$P_{ik} = |V_i|^2 g_{ik} - |V_i||V_k| (g_{ik} \cos \theta_{ik} + b_{ik} \sin \theta_{ik}) ,$$

$$Q_{ik} = -|V_i|^2 b_{ik} - |V_i||V_k| (g_{ik} \sin \theta_{ik} - b_{ik} \cos \theta_{ik}) .$$

Insert $g_{ik} = 0$ and $b_{ik} = -1/X$:

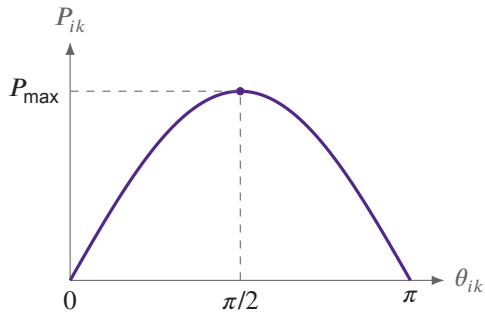
$$P_{ik} = \frac{|V_i||V_k|}{X} \sin \theta_{ik}$$

$$Q_{ik} = \frac{|V_i|^2 - |V_i||V_k| \cos \theta_{ik}}{X}$$

$P_{ki} = -P_{ik}$: the $R = 0$ line has no real power loss.

Maximum real power transfer

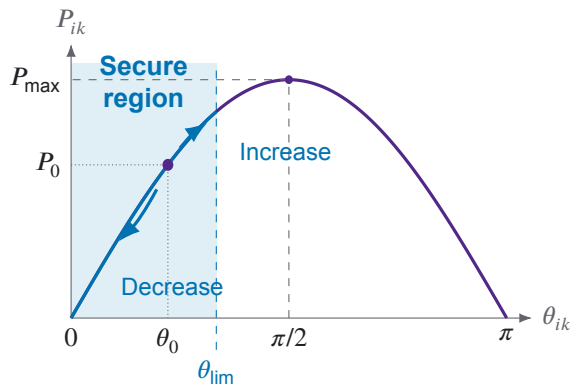
Hold $|V_i|$, $|V_k|$, and $X > 0$ fixed in $P_{ik} = P_{\max} \sin \theta_{ik}$.



$$P_{\max} = \frac{|V_i||V_k|}{X}$$

$$\theta_{ik} = \frac{\pi}{2} \implies P_{ik} = P_{\max}$$

Secure operational region



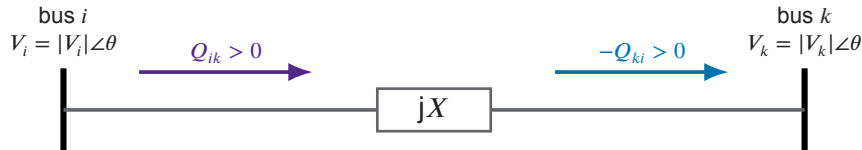
Operating point: (θ_0, P_0)

Illustrative range; limits depend on network conditions and contingencies.

- Limit the angle difference to preserve stability margin.
- Operate inside the secure range, with room to increase or decrease power transfer.
- Normal operation stays away from $\pi/2$, where no upward transfer margin remains.

Reactive power and the magnitude difference

Set $\theta_i = \theta_k = \theta$ and compare magnitudes with $|V_i| > |V_k| > 0$.



At equal angles, the higher-voltage bus supplies Q ; the line also absorbs Q

$$Q_{ik} = \frac{|V_i|(|V_i| - |V_k|)}{X} > 0, \quad Q_{ki} = \frac{|V_k|(|V_k| - |V_i|)}{X} < 0.$$

The reactive power absorbed by the line is

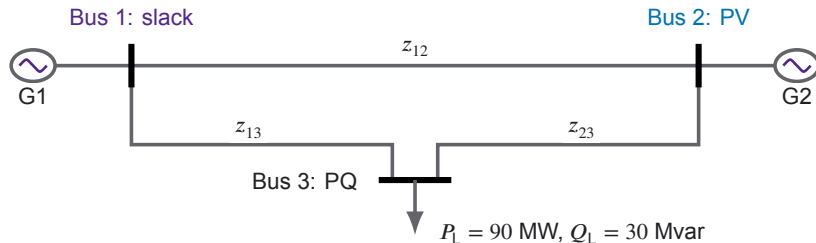
$$Q_{ik} + Q_{ki} = \frac{(|V_i| - |V_k|)^2}{X} > 0.$$

PART 4

Worked example

Present one three-bus system, identify its known and unknown quantities, write the power flow equations, and inspect a numerical solution.

Example: Three-bus system



$$z_{12} = 0.018 + j0.060, \quad z_{13} = 0.070 + j0.210, \quad z_{23} = 0.045 + j0.135.$$

Branch impedances are in p.u. on a 100-MVA base. Unity taps; no charging or shunts.

Example: Known and unknown quantities

All powers and voltage magnitudes below are in per unit.

- **Bus 1: slack**

Known: $|V_1| = 1.03$, $\theta_1 = 0$. Unknown: P_1, Q_1 .

- **Bus 2: PV**

Known: $P_2 = 0.55$, $|V_2| = 1.01$. Unknown: θ_2, Q_2 .

- **Bus 3: PQ**

Known: $P_3 = -0.90$, $Q_3 = -0.30$. Unknown: $|V_3|, \theta_3$.

Buses 1 and 2 have no local load. Solve $\theta_2, \theta_3, |V_3|$, then calculate P_1, Q_1, Q_2 .

Example: Power flow equations

Blue: given quantities

Purple: quantities to determine

$$V_1 = V_1^{\text{spec}}, \quad V_2 = |V_2|^{\text{spec}} e^{j\theta_2}, \quad V_3 = |V_3| e^{j\theta_3}.$$

$$\text{Re}\{V_2 [y_{12}(V_2 - V_1) + y_{23}(V_2 - V_3)]^*\} = P_2^{\text{spec}},$$

$$\text{Re}\{V_3 [y_{13}(V_3 - V_1) + y_{23}(V_3 - V_2)]^*\} = P_3^{\text{spec}},$$

$$\text{Im}\{V_3 [y_{13}(V_3 - V_1) + y_{23}(V_3 - V_2)]^*\} = Q_3^{\text{spec}}.$$

The branch admittances $y_{ik} = 1/z_{ik}$ are known. Solve for $\theta_2, \theta_3, |V_3|$.

Example: Numerical solution

Blue: specified

Purple: calculated

Bus voltages

$$V_1 = 1.03000 \angle 0^\circ$$

$$V_2 = 1.01000 \angle 0.3108^\circ$$

$$V_3 = 0.96441 \angle (-3.6523^\circ)$$

Calculated generator powers

$$P_1 = 37.8855 \text{ MW}$$

$$Q_1 = 54.8871 \text{ Mvar}$$

$$Q_2 = -16.1649 \text{ Mvar}$$

Voltage magnitudes are in p.u.

$Q_2 < 0$: bus 2 absorbs reactive power.

Total generation is $37.8855 + 55 = 92.8855$ MW: 90 MW supplies the load and 2.8855 MW covers line losses.

Summary

- The power equations relate bus voltages to real and reactive injections.
- Bus types identify specified quantities and the unknowns to be determined.
- An inductive line relates real power to angle and reactive power to voltage magnitude.
- A power flow solution supplies the bus state; currents, branch flows, and losses follow from it.

The example follows one route: system and operating data, knowns and unknowns, power flow equations, then numerical results.