Power Flow IV

Gauss–Seidel voltage updates

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Lecture outline

1. Iteration framework: initial voltages, power mismatch, and convergence.
2. Gauss–Seidel updates: PQ, PV, and slack bus treatment.
3. Worked example: one sweep, convergence, and solved outputs.

PART 1

The iteration framework

The bus power equations are nonlinear. Start with voltage estimates, evaluate their power mismatch, and update the voltages until the convergence test is satisfied.

Review: Variables and equations

For a fixed, connected n-bus network, use one slack/reference bus.

- Four real quantities per bus: $P_i, Q_i, |V_i|, \theta_i$.
- The bus type specifies two quantities at each bus.

$$\underbrace{4n}_{\text{bus quantities}} - \underbrace{2n}_{\text{specified}} = \underbrace{2n}_{\text{unknown quantities}}.$$

One complex power relation per bus gives two real equations:

$$P_i + \mathrm{j}Q_i = V_i \left(\sum_{k=1}^{n} Y_{ik} V_k \right)^*, \qquad i = 1, \ldots, n.$$

There are $2n$ unknown quantities and $2n$ real equations (P and Q).

Find the unknown voltages, then calculate the remaining power outputs.

Nonlinear bus power equations

Expand the same complex power relation:

$$P_i + \mathrm{j}Q_i = \sum_{k=1}^{n} Y_{ik}^* V_i V_k^*.$$

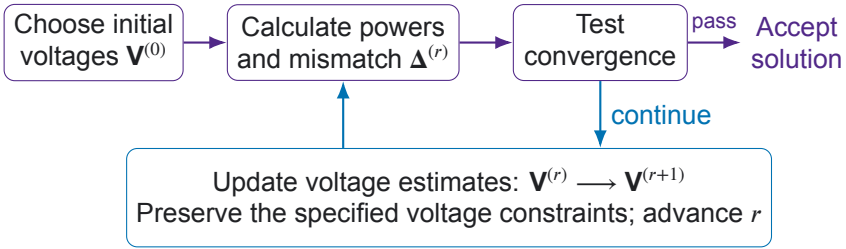
- **Nonlinear:** products of unknown voltages, $V_i V_k^*$.
- **Coupled:** each bus equation contains neighboring voltages.

A single fixed linear solve is insufficient.

Start with voltage estimates; test their power balance and improve them iteratively.

The general iteration loop

Iterate the voltage vector; use power mismatch to test each estimate.



Mismatch tests the estimate; an update rule produces the next voltage vector.

Choose initial voltages

Keep the network and operating specifications fixed.

$$\mathbf{V}^{(0)} = \begin{bmatrix} V_1^{(0)} & \cdots & V_n^{(0)} \end{bmatrix}^{\mathsf{T}}, \qquad V_i^{(0)} = |V_i|^{(0)} e^{\mathrm{j}\theta_i^{(0)}}.$$

- Retain every specified voltage magnitude and angle.
- Guess the remaining voltage components; use nonzero magnitudes.
- A simple start: 1 p.u. and 0° for unspecified magnitudes and angles.

The initial voltage vector gives a trial network state. Calculate its powers next.

Calculate the power mismatch

At iteration r, calculate powers from the same voltage vector:

$$S_i^{\text{calc}}(\mathbf{V}^{(r)}) = V_i^{(r)} \left(\sum_{k=1}^{n} Y_{ik} V_k^{(r)} \right)^* = P_i^{\text{calc}} + \text{j} Q_i^{\text{calc}}.$$

Compare each specified power with its calculated value:

$$\Delta P_i^{(r)} = P_i^{\text{spec}} - P_i^{\text{calc}}(\mathbf{V}^{(r)}) \quad \text{where P is specified,}$$

$$\Delta Q_i^{(r)} = Q_i^{\text{spec}} - Q_i^{\text{calc}}(\mathbf{V}^{(r)}) \quad \text{where Q is specified.}$$

Collect these residuals into $\boldsymbol{\Delta}^{(r)}$.

Changing the voltage estimate changes the calculated powers and their mismatch.

Test convergence and continue

Test the largest absolute power mismatch:

$$\|\boldsymbol{\Delta}^{(r)}\|_{\infty} \leq \varepsilon.$$

- **Pass:** accept with the active operating constraints satisfied.
- **Continue:** update voltages, recalculate powers, and test again.
- **Failure:** report nonconvergence at an iteration cap or failed update.

After convergence, calculate the remaining bus powers and branch flows.

The loop is defined. Next, derive the voltage update for each bus type.

PART 2

Gauss–Seidel bus updates

Recall the known and unknown quantities at each bus, then derive its Gauss–Seidel treatment.

Review: Bus specifications and unknowns

P_i, Q_i are net injections: generation minus local load.

- **PQ bus**
  Known: P_i, Q_i Unknown: $|V_i|, \theta_i$.
- **PV bus**
  Known: $P_i, |V_i|$ Unknown: θ_i, Q_i.
- **Slack bus**
  Known: $|V_i|, \theta_i$ Unknown: P_i, Q_i.

PQ: update magnitude and angle. PV: update angle. Slack: fixed voltage.

PQ bus: Power equation to current

Given P and Q, update the complete complex voltage:

$$P_i^{\text{spec}} + \text{j}Q_i^{\text{spec}} = V_i \left(\sum_{k=1}^{n} Y_{ik} V_k \right)^* .$$

Take the complex conjugate of both sides:

$$P_i^{\text{spec}} - \text{j}Q_i^{\text{spec}} = V_i^* \sum_{k=1}^{n} Y_{ik} V_k .$$

Divide by $V_i^* \neq 0$:

$$\boxed{\sum_{k=1}^{n} Y_{ik} V_k = \frac{P_i^{\text{spec}} - \text{j}Q_i^{\text{spec}}}{V_i^*} .}$$

PQ bus: Isolate the voltage

Separate the diagonal term from the sum:

$$Y_{ii}V_i + \sum_{k \neq i} Y_{ik}V_k = \frac{P_i^{\text{spec}} - \text{j}Q_i^{\text{spec}}}{V_i^*}.$$

Move the neighbor terms and divide by $Y_{ii} \neq 0$:

$$\boxed{V_i = \frac{1}{Y_{ii}} \left[\frac{P_i^{\text{spec}} - \text{j}Q_i^{\text{spec}}}{V_i^*} - \sum_{k \neq i} Y_{ik}V_k \right].}$$

This is still implicit: V_i^* and the neighboring voltages remain on the right.

Use stored estimates on the right to calculate a new voltage on the left.

PQ bus: Gauss–Seidel update

In sweep $r+1$, update PQ bus i; read its neighboring voltages $\widehat{V}_k$.

$$V_i^{(r+1)} = \frac{1}{Y_{ii}} \left[\frac{P_i^{\text{spec}} - \mathrm{j} Q_i^{\text{spec}}}{(V_i^{(r)})^*} - \sum_{k \neq i} Y_{ik} \widehat{V}_k \right].$$

$$\widehat{V}_k = \begin{cases} V_k^{(r+1)} & \text{updated non-slack neighbor,} \\ V_k^{(r)} & \text{unvisited non-slack neighbor,} \\ V_k^{\text{spec}} & \text{slack neighbor (fixed).} \end{cases}$$

Store and reuse the new voltage. Check both P and Q after the complete sweep.

PV bus: Fixed magnitude, unknown angle

$P_i, |V_i|^{\text{spec}}$ are fixed; only the voltage angle θ_i is updated.

Estimate the net reactive injection (est = current estimate):

$$I_i^{\text{est}} = Y_{ii} V_i^{(r)} + \sum_{k \neq i} Y_{ik} \widehat{V}_k,$$

$$Q_i^{\text{est}} = \text{Im}\left\{ V_i^{(r)} (I_i^{\text{est}})^* \right\}.$$

Net injection = generation minus local load ($Q_{\text{L},i} > 0$ for consumption).

$$\underbrace{Q_{\text{G},i}^{\text{est}}}_{\text{generator output}} = \underbrace{Q_i^{\text{est}}}_{\text{net injection}} + \underbrace{Q_{\text{L},i}}_{\text{local load}}.$$

Use Q_i^{est} in the voltage update; check generator limits on $Q_{\text{G},i}^{\text{est}}$.

PV bus: Calculate the new angle

Within generator Q limits, use the estimated net injection:

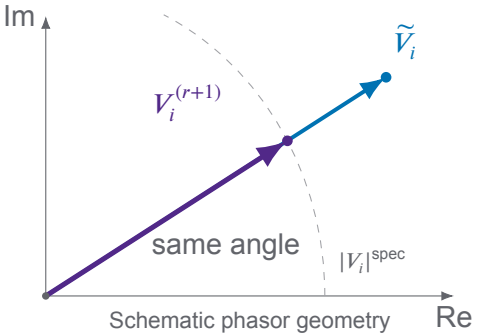
$$\widetilde{V}_i = \frac{1}{Y_{ii}} \left[\frac{P_i^{\text{spec}} - \text{j}Q_i^{\text{est}}}{(V_i^{(r)})^*} - \sum_{k \neq i} Y_{ik} \widehat{V}_k \right].$$

Retain its angle:

$$\boxed{\theta_i^{(r+1)} = \arg(\widetilde{V}_i).}$$

Combine the new angle with the prescribed magnitude before storing the voltage.

PV bus: Keep magnitude, update angle



$$V_i^{(r+1)} = |V_i|^{\text{spec}} e^{j\theta_i^{(r+1)}}$$

$$= |V_i|^{\text{spec}} \frac{\widetilde{V}_i}{|\widetilde{V}_i|}.$$

The new phasor keeps the temporary angle and lies on the circle of specified radius.

Only the angle changes; the complex voltage changes with it.

Store the projected voltage; test P after the sweep. Q remains an output.

PV to PQ at a reactive power limit

If voltage control requires Q outside the generator limits, enforce the bound:

$$Q_{\mathrm{G},i} = Q_{\mathrm{G},i}^{\lim}, \qquad Q_i^{\mathrm{spec}} = Q_{\mathrm{G},i}^{\lim} - Q_{\mathrm{L},i}.$$

PV bus

Specified: $P_i,\ |V_i|$

Unknown: $\theta_i,\ Q_i$

$\Longrightarrow$

PQ bus

Specified: $P_i,\ Q_i^{\mathrm{spec}}$

Unknown: $\theta_i,\ |V_i|$

Use the PQ update; solve the magnitude instead of restoring its former setpoint.

Slack bus: Keep the voltage fixed

The voltage magnitude and reference angle are specified:

$$\boxed{V_s^{(r+1)} = V_s^{(r)} = V_s^{\text{spec}}.}$$

- Use this fixed phasor in every neighboring bus update.
- Slack P and Q are outputs; neither enters the mismatch vector.

$$P_s + \mathrm{j}Q_s = V_s^{\text{spec}} \left(\sum_{k=1}^{n} Y_{sk} V_k \right)^*.$$

After the full sweep, return to the power mismatch test in the general loop.

PART 3

Worked example

Apply the general update to one slack, one PV, and one PQ bus. Write the symbolic updates first, follow one complete sweep, and then inspect the numerical convergence.

Example: Generators, load, and bus types

100-MVA base. Branch impedances are in p.u.

Bus 1: Slack generator

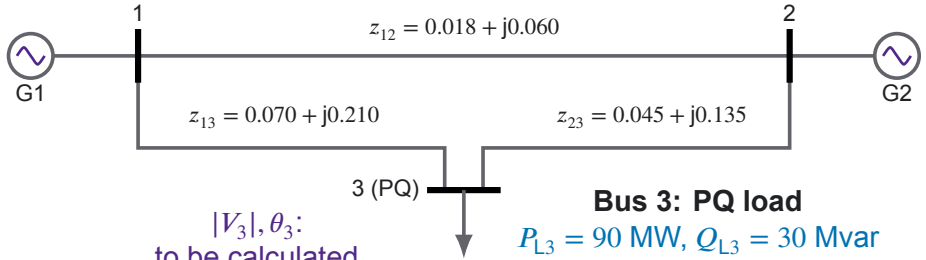
$V_1 = 1.03\angle 0^\circ$ p.u.

P_{G1}, Q_{G1}: to be calculated

Bus 2: PV generator

$P_{G2} = 55$ MW, $|V_2| = 1.01$ p.u.

Q_{G2}, θ_2: to be calculated



Bus 3: PQ load

$P_{L3} = 90$ MW, $Q_{L3} = 30$ Mvar

$P_{G3} = Q_{G3} = 0$

No loads at buses 1 or 2. No charging or shunts; unity taps.

Example: Specified injections and unknowns

Net injections (p.u.): $P_i = P_{\mathrm{G},i} - P_{\mathrm{L},i}$, $Q_i = Q_{\mathrm{G},i} - Q_{\mathrm{L},i}$.

- **Bus 1 (slack):** $V_1 = 1.03\angle 0^\circ$.
  Unknown: P_1, Q_1; these equal the generator outputs.
- **Bus 2 (PV):** $P_2 = 55/100 = 0.55$, $|V_2| = 1.01$.
  Unknown: θ_2, Q_2; $-0.25 \le Q_{\mathrm{G2}} = Q_2 \le 0.50$ p.u.
- **Bus 3 (PQ):** $P_3 = -90/100 = -0.90$, $Q_3 = -30/100 = -0.30$.
  Unknown: $|V_3|, \theta_3$.

Form Y-bus from $y_{ik} = 1/z_{ik}$**:** $Y_{ik} = -y_{ik}$ **(**$i \neq k$**),** $Y_{ii} = \sum_{k \neq i} y_{ik}$.

Example: Symbolic PQ update

Visit bus 3 (PQ), then bus 2 (PV). Bus 1 stays fixed.

Blue: given Purple: computed in this sweep

$$V_3^{(r+1)} = \frac{1}{Y_{33}} \left[\frac{P_3^{\text{spec}} - \mathrm{j} Q_3^{\text{spec}}}{(V_3^{(r)})^*} - Y_{31} V_1 - Y_{32} V_2^{(r)} \right].$$

- Bus 1 supplies its fixed slack voltage.
- Bus 2 has not yet been updated: use $V_2^{(r)}$.

Store the complete new V_3; the PV visit uses it immediately.

Example: Symbolic PV update

At bus 2 (PV), use the new $V_3^{(r+1)}$ and retain $|V_2|^{\text{spec}}$.

$$Q_2^{\text{est}} = \text{Im}\left\{ V_2^{(r)} \left(Y_{21}V_1 + Y_{22}V_2^{(r)} + Y_{23}V_3^{(r+1)} \right)^* \right\}.$$

Within Q limits, form the temporary voltage:

$$\widetilde{V}_2 = \frac{1}{Y_{22}} \left[\frac{P_2^{\text{spec}} - \text{j}Q_2^{\text{est}}}{(V_2^{(r)})^*} - Y_{21}V_1 - Y_{23}V_3^{(r+1)} \right].$$

$$\theta_2^{(r+1)} = \arg \widetilde{V}_2, \qquad V_2^{(r+1)} = |V_2|^{\text{spec}} e^{\text{j}\theta_2^{(r+1)}}.$$

The angle changes; the PV magnitude remains specified.

Example: Initial state and mismatch

$$\mathbf{V}^{(0)} = [1.03, 1.01, 1]^{\mathsf{T}}, \qquad \boldsymbol{\Delta} = [\Delta P_2, \Delta P_3, \Delta Q_3]^{\mathsf{T}}.$$

Calculate $\mathbf{I}^{(0)} = \mathbf{Y}_{\text{bus}}\mathbf{V}^{(0)}$ and $S_i^{(0)} = V_i^{(0)} I_i^{(0)*}$:

$$\boldsymbol{\Delta}^{(0)} = \begin{bmatrix} 0.55 \\ -0.90 \\ -0.30 \end{bmatrix} - \begin{bmatrix} -0.070216106 \\ -0.065079365 \\ -0.195238095 \end{bmatrix} = \begin{bmatrix} 0.620216106 \\ -0.834920635 \\ -0.104761905 \end{bmatrix}.$$

- Visit PQ bus 3, then PV bus 2; no relaxation.
- Stop at $\|\boldsymbol{\Delta}\|_\infty \le 10^{-8}$ p.u.; cap: 100 sweeps.

The initial norm is 0.834920635 **p.u. A voltage update is needed.**

Example: PQ voltage update

$$V_3^{(1)} = \frac{1}{Y_{33}} \left[\frac{-0.90 + \text{j}0.30}{(V_3^{(0)})^*} - Y_{31}V_1 - Y_{32}V_2^{(0)} \right].$$

At this visit, $V_3^{(0)} = 1$ and both neighbors retain their initial values.

$$Y_{31}(1.03) + Y_{32}(1.01) = -3.715873016 + \text{j}11.147619048$$

$$V_3^{(1)} = \frac{2.815873016 - \text{j}10.847619048}{3.650793651 - \text{j}10.952380952} = 0.968521739 - \text{j}0.065739130.$$

Store the new $V_3^{(1)}$ and use it immediately in the PV visit.

Example: PV reactive estimate

Use the new $V_3^{(1)}$ and the stored $V_2^{(0)} = 1.01$:

$$Q_2^{\text{est}} = \text{Im}\left\{ 1.01 \left[Y_{21}(1.03) + Y_{22}(1.01) + Y_{23} V_3^{(1)} \right]^* \right\}.$$

At bus 2, $Q_{\text{L},2} = 0$, so

$$Q_{\text{G2}}^{\text{est}} = Q_2^{\text{est}} = -0.177129371 \text{ p.u..}$$

$$-0.25 \leq -0.177129371 \leq 0.50.$$

The Q estimate is within the generator limits. Keep $|V_2| = 1.01$ and update its angle.

Example: PV angle and stored voltage

Insert the reactive estimate into the temporary complex update:

$$\widetilde{V}_2 = \frac{1}{Y_{22}}\left[\frac{0.55 + \mathrm{j}0.177129371}{1.01} - Y_{21}(1.03) - Y_{23}V_3^{(1)}\right].$$

$$\widetilde{V}_2 = 1.011364010 + \mathrm{j}0.004398320, \qquad \theta_2^{(1)} = \arg \widetilde{V}_2.$$

Store the new angle with the fixed magnitude:

$$V_2^{(1)} = 1.01 e^{\mathrm{j}\theta_2^{(1)}} = \frac{1.01}{|\widetilde{V}_2|}\widetilde{V}_2 = \boxed{1.009990449 + \mathrm{j}0.004392346}.$$

The sweep is complete. Recalculate power mismatch from the stored voltage vector.

Example: Mismatch after one sweep

The completed vector is $\mathbf{V}^{(1)} = [V_1, V_2^{(1)}, V_3^{(1)}]^{\mathsf{T}}$. Recalculate all powers from this same vector:

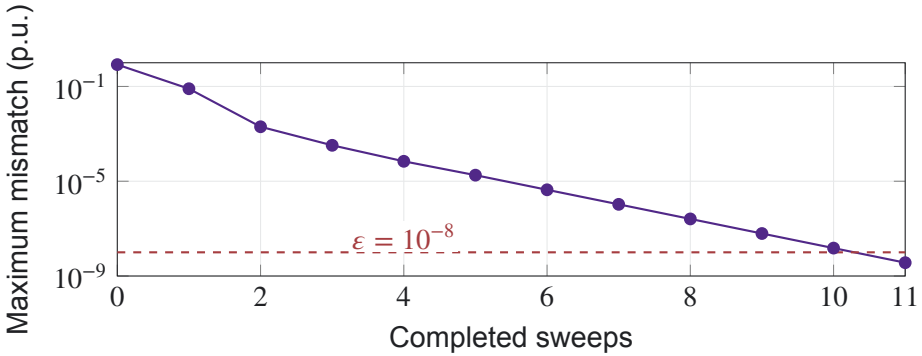
Row	Specified	Calculated	Mismatch
P_2	0.550000000	0.541318393	+0.008681607
P_3	−0.900000000	−0.919085452	+0.019085452
Q_3	−0.300000000	−0.219952520	−0.080047480

$$\|\boldsymbol{\Delta}^{(1)}\|_\infty = 0.080047480 > 10^{-8}$$

After the sweep, $Q_2 = -0.205197242$ p.u. It differs from the intermediate estimate and remains a calculated output.

The Q mismatch at bus 3 controls this norm. Continue with another complete sweep.

Example: Repeated sweeps



Sweep 10: 1.50701×10^{-8}. **Sweep 11:** 3.65317×10^{-9} **p.u., the first passing state.**

Example: Calculate the slack output

Use the accepted voltages after sweep 11 (p.u.):

$$V_1 = 1.03\angle 0^\circ, \quad V_2 \approx 1.01\angle 0.310813^\circ, \quad V_3 \approx 0.964414\angle(-3.652309^\circ).$$

$$\begin{aligned} I_1 &= Y_{11}V_1 + Y_{12}V_2 + Y_{13}V_3 \\ &\approx 0.367820175 - \mathrm{j}0.532884938. \end{aligned}$$

$$S_1 = V_1 I_1^* \approx 0.378854780 + \mathrm{j}0.548871486 \text{ p.u.}.$$

No local load at bus 1; on the 100-MVA base:

$$\boxed{P_{G1} = 37.88548 \text{ MW}, \qquad Q_{G1} = 54.88715 \text{ Mvar}.}$$

Slack voltage stays fixed; its real and reactive outputs follow from the solved network.

Example: Solved voltages and outputs

Blue: given data Purple: unknown or calculated quantities

Bus	$\lvert V \rvert$ (p.u.)	Angle (deg.)	Net P (MW)	Net Q (Mvar)
1	1.030000	0	37.88548	54.88715
2	1.010000	0.310813	55.00000	-16.16487
3	0.964414	-3.652309	-90.00000	-30.00000

$$\|\boldsymbol{\Delta}^{(11)}\|_\infty = 3.65317 \times 10^{-9} < 10^{-8} \text{ p.u..}$$

Bus 2 Q is within $[-25, 50]$ Mvar, so the PV constraint is consistent.

The solved voltages determine the remaining powers. Retain full precision when calculating outputs and testing mismatch.

Summary

1. The PQ update follows by isolating $Y_{ii}V_i$ and substituting stored voltages.
2. PV control adds a reactive estimate and a magnitude constraint, subject to capability.
3. A sweep stores each new voltage immediately and then tests all active power balances.
4. The accepted voltage vector satisfies the power targets and determines the remaining outputs.

Next, use the same network, start, and tolerance to obtain coupled voltage corrections with Newton–Raphson.