Power Flow VI

Newton–Raphson worked example

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Lecture outline

1. Quick review: scalar Newton, vector Newton, and power flow Newton.
2. Worked example: two complete Newton corrections on one network.

We will track the same three unknowns, three power equations, and tolerance throughout.

> **Every numerical result will connect to the equation, state, and operation that produced it.**

PART 1

Quick review

Recall the scalar and vector Newton updates, then apply the same zero-residual form to power flow.

Review: Scalar Newton iteration

Find a root of $f(x) = 0$ using the local slope:

$$\boxed{\Delta x = -\frac{f(x^{(r)})}{f'(x^{(r)})}}, \qquad f'(x^{(r)}) \neq 0.$$

$$\boxed{x^{(r+1)} = x^{(r)} + \Delta x}.$$

The tangent gives the correction. Check the original f at the updated x.

Review: Vector Newton iteration

For $\mathbf{f}(\mathbf{x}) = \mathbf{0}$, collect the local derivatives into the Jacobian:

$$J_{ij} = \frac{\partial f_i}{\partial x_j}.$$

At the current state, solve and update:

$$\boxed{\mathbf{J}(\mathbf{x}^{(r)})\Delta\mathbf{x} = -\mathbf{f}(\mathbf{x}^{(r)})}, \qquad \boxed{\mathbf{x}^{(r+1)} = \mathbf{x}^{(r)} + \Delta\mathbf{x}}.$$

Reevaluate the original residuals after every update. Use a linear solve for the correction.

Review: Power flow in zero-residual form

As in Lecture 36, move every term in the power balance to the left:

$$\boxed{f_{Pi}(\mathbf{x}) = P_i(\mathbf{x}) + P_{Di} - P_{Gi} = 0},$$

$$\boxed{f_{Qi}(\mathbf{x}) = Q_i(\mathbf{x}) + Q_{Di} - Q_{Gi} = 0}.$$

$P_i(\mathbf{x})$ and $Q_i(\mathbf{x})$ are network injections evaluated from the trial voltages. G denotes generation; D denotes demand.

A Newton step corrects the voltage state to drive these residuals toward zero.

Review: Selected unknowns and residuals

Let A be the non-slack buses and L the PQ buses.

$$\mathbf{x} = \begin{bmatrix} \theta_A \\ \mathbf{v}_L \end{bmatrix}, \qquad \mathbf{f}(\mathbf{x}) = \begin{bmatrix} \left(f_{Pi}(\mathbf{x})\right)_{i \in A} \\ \left(f_{Qi}(\mathbf{x})\right)_{i \in L} \end{bmatrix} = \mathbf{0}.$$

Bus	Voltage unknowns	Residual rows
PQ	θ_i, v_i	f_{Pi}, f_{Qi}
PV	θ_i	f_{Pi}
Slack	None	None in this solve

Keep the slack phasor and PV magnitudes fixed. Use the same row and column order throughout.

Review: Power flow Newton update

The same Newton equation now has four derivative blocks:

$$\underbrace{\begin{bmatrix} \mathbf{H} & \mathbf{N} \\ \mathbf{M} & \mathbf{L} \end{bmatrix}^{(r)}}_{\mathbf{J}=\partial\mathbf{f}/\partial\mathbf{x}} \begin{bmatrix} \Delta\theta_A \\ \Delta\mathbf{v}_L \end{bmatrix} = - \begin{bmatrix} \mathbf{f}_P(\mathbf{x}^{(r)}) \\ \mathbf{f}_Q(\mathbf{x}^{(r)}) \end{bmatrix}.$$

Add the correction and test the original residuals:

$$\mathbf{x}^{(r+1)} = \mathbf{x}^{(r)} + \Delta\mathbf{x}, \qquad \|\mathbf{f}(\mathbf{x}^{(r+1)})\|_\infty \le \varepsilon.$$

If the test fails, rebuild **J** at the new state and repeat.

Detailed expressions: see Lecture 36, Jacobian derivation.

PART 2

The complete worked example

Follow one three-bus system through every power calculation, derivative, linear solve, voltage update, and residual check.

Example: Generators, load, and network

100-MVA base. Branch impedances are in p.u.

Bus 1: Slack generator

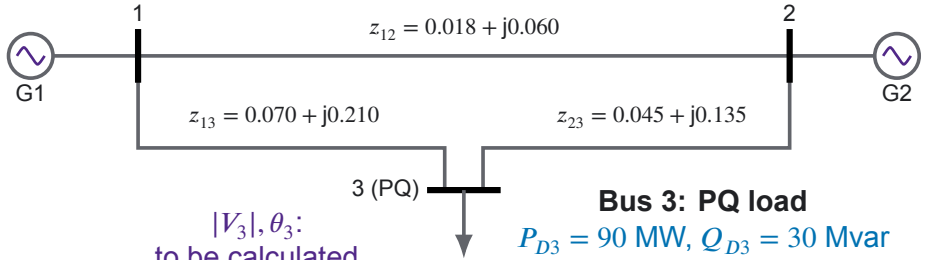
$V_1 = 1.03\angle 0^\circ$ p.u.

P_{G1}, Q_{G1}: to be calculated

Bus 2: PV generator

$P_{G2} = 55$ MW, $|V_2| = 1.01$ p.u.

Q_{G2}, θ_2: to be calculated



$|V_3|, \theta_3$:
to be calculated

Bus 3: PQ load

$P_{D3} = 90$ MW, $Q_{D3} = 30$ Mvar

$P_{G3} = Q_{G3} = 0$

No loads at buses 1 or 2. No charging or shunts. Unity taps.

Example: Specified data and unknowns

Bus	Specified	To calculate
1: slack	$V_1 = 1.03\angle 0^\circ$	P_1, Q_1 after solving
2: PV	$P_{G2} = 0.55, \quad v_2 = 1.01$	θ_2 and dependent Q_2
3: PQ	$P_{D3} = 0.90, \quad Q_{D3} = 0.30$	θ_3, v_3

Demand enters the residual with a plus sign; generation enters with a minus sign. Powers are in p.u.

Bus 2: $-0.25 \le Q_{G2} = Q_2 \le 0.50$ p.u.

Only θ_2, θ_3, v_3 enter the Newton state. The other voltages remain fixed.

Network calculation: Branch admittances

$$y_{ik} = \frac{1}{r_{ik} + \mathrm{j}x_{ik}} = \frac{r_{ik} - \mathrm{j}x_{ik}}{r_{ik}^2 + x_{ik}^2}.$$

$$y_{12} = \frac{1}{0.018 + \mathrm{j}0.060} = 4.587156 - \mathrm{j}15.290520$$

$$y_{13} = \frac{1}{0.070 + \mathrm{j}0.210} = 1.428571 - \mathrm{j}4.285714$$

$$y_{23} = \frac{1}{0.045 + \mathrm{j}0.135} = 2.222222 - \mathrm{j}6.666667.$$

Use these admittances to form Y-bus. All subsequent displayed numbers are rounded.

Network calculation: The complete Y-bus

$Y_{ii} = \sum_{k \neq i} y_{ik}$ and $Y_{ik} = -y_{ik}$ for $i \neq k$. Write $\mathbf{Y} = \mathbf{G} + \mathrm{j}\mathbf{B}$.

$$\mathbf{G} = \begin{bmatrix} 6.015727 & -4.587156 & -1.428571 \\ -4.587156 & 6.809378 & -2.222222 \\ -1.428571 & -2.222222 & 3.650794 \end{bmatrix}$$

$$\mathbf{B} = \begin{bmatrix} -19.576234 & 15.290520 & 4.285714 \\ 15.290520 & -21.957187 & 6.666667 \\ 4.285714 & 6.666667 & -10.952381 \end{bmatrix}$$

Y-bus remains fixed throughout this example. The Jacobian changes with the voltage state.

Initial guess and residual order

$$\mathbf{x} = \begin{bmatrix} \theta_2 \\ \theta_3 \\ v_3 \end{bmatrix}, \qquad \mathbf{x}^{(0)} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \qquad \mathbf{f}(\mathbf{x}) = \begin{bmatrix} P_2(\mathbf{x}) - 0.55 \\ P_3(\mathbf{x}) + 0.90 \\ Q_3(\mathbf{x}) + 0.30 \end{bmatrix} = \mathbf{0}.$$

$$\mathbf{V}^{(0)} = \begin{bmatrix} 1.03\angle 0^\circ \\ 1.01\angle 0^\circ \\ 1.00\angle 0^\circ \end{bmatrix}.$$

Residual rows: f_{P2}, f_{P3}, f_{Q3}. State columns: θ_2, θ_3, v_3.

Teaching tolerance: $\varepsilon = 10^{-3}$ p.u. Accept at $\|\mathbf{f}\|_\infty \le \varepsilon$. Full Newton steps; cap: 100 corrections.

> **On the 100-MVA base, this tolerance is 0.1 MW or Mvar per equation. Use radians for angles.**

The network functions inside the residuals

Let $v_i = |V_i|$ and $\theta_{ik} = \theta_i - \theta_k$. Write P_i and Q_i for the network functions at the current voltage:

$$\begin{aligned} P_2 &= G_{22}v_2^2 + v_2v_1(G_{21}\cos\theta_{21} + B_{21}\sin\theta_{21}) \\ &\quad + v_2v_3(G_{23}\cos\theta_{23} + B_{23}\sin\theta_{23}), \\ P_3 &= G_{33}v_3^2 + v_3v_1(G_{31}\cos\theta_{31} + B_{31}\sin\theta_{31}) \\ &\quad + v_3v_2(G_{32}\cos\theta_{32} + B_{32}\sin\theta_{32}). \end{aligned}$$

$$Q_3 = -B_{33}v_3^2 + v_3v_1(G_{31}\sin\theta_{31} - B_{31}\cos\theta_{31}) + v_3v_2(G_{32}\sin\theta_{32} - B_{32}\cos\theta_{32}).$$

Use these functions in $f_{P2} = P_2 - 0.55$, $f_{P3} = P_3 + 0.90$, $f_{Q3} = Q_3 + 0.30$.

Iteration 0: Calculate P2 and P3

At the initial state, every angle difference is zero: $\cos 0 = 1$ and $\sin 0 = 0$.

$$
\begin{aligned}
P_2^{(0)} &= 1.01\big[(-4.587155963)(1.03) \\
&\qquad + (6.809378186)(1.01) + (-2.222222222)(1)\big] \\
&= -0.070216106,
\end{aligned}
$$

$$
\begin{aligned}
P_3^{(0)} &= 1\big[(-1.428571429)(1.03) \\
&\qquad + (-2.222222222)(1.01) + (3.650793651)(1)\big] \\
&= -0.065079365.
\end{aligned}
$$

These are powers implied by the initial voltages, before the schedules are satisfied.

Iteration 0: Calculate Q3 and Q2

At zero angle differences, $Q_i = -v_i \sum_k B_{ik} v_k$.

$$
\begin{aligned}
Q_3^{(0)} &= -1\big[(4.285714286)(1.03) \\
&\qquad + (6.666666667)(1.01) + (-10.952380952)(1)\big] \\
&= -0.195238095,
\end{aligned}
$$

$$
\begin{aligned}
Q_2^{(0)} &= -1.01\big[(15.290519878)(1.03) \\
&\qquad + (-21.957186544)(1.01) + (6.666666667)(1)\big] \\
&= -0.241535168.
\end{aligned}
$$

Q3 is a selected equation. Q2 is needed for a diagonal derivative and the PV capability check.

Iteration 0: Evaluate the residual

Substitute the initial powers into the same zero-residual equations:

$$\mathbf{f}(\mathbf{x}^{(0)}) = \begin{bmatrix} P_2^{(0)} - 0.55 \\ P_3^{(0)} + 0.90 \\ Q_3^{(0)} + 0.30 \end{bmatrix} = \begin{bmatrix} -0.070216106 - 0.55 \\ -0.065079365 + 0.90 \\ -0.195238095 + 0.30 \end{bmatrix} = \begin{bmatrix} -0.620216106 \\ 0.834920635 \\ 0.104761905 \end{bmatrix}.$$

$$\|\mathbf{f}(\mathbf{x}^{(0)})\|_\infty = 0.834920635 > 10^{-3}.$$

The positive f_{P3} means $P_3 + 0.90 > 0$: the network injection must become more negative.

The residual is not zero. The Newton correction solves $\mathbf{J}\Delta\mathbf{x} = -\mathbf{f}$.

Iteration 0: Label the Jacobian

$$\mathbf{J} = \frac{\partial \mathbf{f}}{\partial \mathbf{x}} = \begin{bmatrix} \partial P_2/\partial \theta_2 & \partial P_2/\partial \theta_3 & \partial P_2/\partial v_3 \\ \partial P_3/\partial \theta_2 & \partial P_3/\partial \theta_3 & \partial P_3/\partial v_3 \\ \partial Q_3/\partial \theta_2 & \partial Q_3/\partial \theta_3 & \partial Q_3/\partial v_3 \end{bmatrix} = \begin{bmatrix} H_{22} & H_{23} & N_{23} \\ H_{32} & H_{33} & N_{33} \\ M_{32} & M_{33} & L_{33} \end{bmatrix}.$$

Use the expressions derived in Lecture 36. Substitute the bus indices, then the current voltages and Y-bus entries.

At the initial state, $\theta_2 - \theta_3 = \theta_3 - \theta_2 = 0$, so $\sin 0 = 0$ and $\cos 0 = 1$.

The fixed injection terms have zero derivatives. These P/Q derivatives are the derivatives of our residuals.

Iteration 0: The P2 derivative row

$$
\begin{aligned}
H_{22} &= -Q_2 - B_{22}v_2^2 \\
&= -(-0.241535168) - (-21.957186544)(1.01)^2 = 22.640061
\end{aligned}
$$

$$
\begin{aligned}
H_{23} &= v_2 v_3 \left[G_{23}\sin(\theta_2 - \theta_3) - B_{23}\cos(\theta_2 - \theta_3)\right] \\
&= (1.01)(1)\left[-2.222222222\sin(0) - 6.666666667\cos(0)\right] \\
&= -6.733333
\end{aligned}
$$

$$
\begin{aligned}
N_{23} &= v_2 \left[G_{23}\cos(\theta_2 - \theta_3) + B_{23}\sin(\theta_2 - \theta_3)\right] \\
&= (1.01)\left[-2.222222222\cos(0) + 6.666666667\sin(0)\right] = -2.244444.
\end{aligned}
$$

Iteration 0: The P3 derivative row

$$
\begin{aligned}
H_{32} &= v_3 v_2 \left[G_{32} \sin(\theta_3 - \theta_2) - B_{32} \cos(\theta_3 - \theta_2) \right] \\
&= (1)(1.01) \left[-2.222222222 \sin(0) - 6.666666667 \cos(0) \right] \\
&= -6.733333
\end{aligned}
$$

$$
\begin{aligned}
H_{33} &= -Q_3 - B_{33} v_3^2 \\
&= -(-0.195238095) - (-10.952380952)(1)^2 = 11.147619
\end{aligned}
$$

$$
\begin{aligned}
N_{33} &= \frac{P_3}{v_3} + G_{33} v_3 \\
&= \frac{-0.065079365}{1} + (3.650793651)(1) = 3.585714.
\end{aligned}
$$

Iteration 0: The Q3 derivative row

$$
\begin{aligned}
M_{32} &= -v_3 v_2 \left[G_{32} \cos(\theta_3 - \theta_2) + B_{32} \sin(\theta_3 - \theta_2) \right] \\
&= -(1)(1.01) \left[-2.222222222 \cos(0) + 6.666666667 \sin(0) \right] \\
&= 2.244444
\end{aligned}
$$

$$
\begin{aligned}
M_{33} &= P_3 - G_{33} v_3^2 \\
&= -0.065079365 - (3.650793651)(1)^2 = -3.715873
\end{aligned}
$$

$$
\begin{aligned}
L_{33} &= \frac{Q_3}{v_3} - B_{33} v_3 \\
&= \frac{-0.195238095}{1} - (-10.952380952)(1) = 10.757143.
\end{aligned}
$$

Iteration 0: Solve for the correction

$$\underbrace{\begin{bmatrix} 22.640061 & -6.733333 & -2.244444 \\ -6.733333 & 11.147619 & 3.585714 \\ 2.244444 & -3.715873 & 10.757143 \end{bmatrix}}_{\mathbf{J}^{(0)}} \begin{bmatrix} \Delta\theta_2 \\ \Delta\theta_3 \\ \Delta v_3 \end{bmatrix} = -\underbrace{\begin{bmatrix} -0.620216106 \\ 0.834920635 \\ 0.104761905 \end{bmatrix}}_{\mathbf{f}(\mathbf{x}^{(0)})}.$$

Multiply on the left by $(\mathbf{J}^{(0)})^{-1}$:

$$\boxed{\Delta\mathbf{x}^{(0)} = -(\mathbf{J}^{(0)})^{-1}\mathbf{f}(\mathbf{x}^{(0)})} = \begin{bmatrix} +0.006105173 \\ -0.060900164 \\ -0.032049579 \end{bmatrix}.$$

The entries are $\Delta\theta_2$, $\Delta\theta_3$, and Δv_3, in that order.

These are voltage corrections. Add them to the current state on the next page.

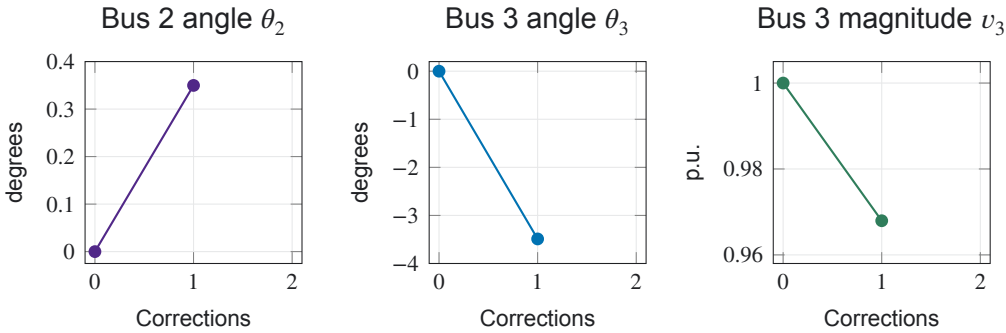
First correction: Update the voltage state

$$\mathbf{x}^{(1)} = \mathbf{x}^{(0)} + \Delta\mathbf{x}^{(0)} = \begin{bmatrix} 0.000 \\ 0.000 \\ 1.000 \end{bmatrix} + \begin{bmatrix} 0.006105173 \\ -0.060900164 \\ -0.032049579 \end{bmatrix} = \begin{bmatrix} 0.006105173 \\ -0.060900164 \\ 0.967950421 \end{bmatrix}.$$

$$V_2^{(1)} = 1.01\angle(0.349801^\circ), \qquad V_3^{(1)} = 0.967950421\angle(-3.489322^\circ).$$

Keep V1 and v2 fixed. The new state is not yet an accepted solution.

Voltage changes after correction 1



$$\mathbf{x}^{(1)} = \begin{bmatrix} 0.006105173 & -0.060900164 & 0.967950421 \end{bmatrix}^{\mathsf{T}}.$$

The plots show degrees for angles. The state vector below uses radians and per unit.

What the first linear model predicts

Each row adds the effects of all three voltage corrections.

Row	$J_{i1}\Delta\theta_2$	$J_{i2}\Delta\theta_3$	$J_{i3}\Delta v_3$	Sum
f_{P2}	0.138221	0.410061	0.071934	0.620216
f_{P3}	-0.041108	-0.678892	-0.114921	-0.834921
f_{Q3}	0.013703	0.226297	-0.344762	-0.104762

$$\mathbf{f}(\mathbf{x}^{(0)}) + \mathbf{J}^{(0)}\Delta\mathbf{x}^{(0)} = \mathbf{0}.$$

The linear model predicts zero residual. The next check uses the original nonlinear equations.

At state 1: Recalculate P2

$$\theta_{21} = 0.006105173, \qquad \theta_{23} = 0.067005338, \qquad v_3 = 0.967950421.$$

$$\begin{aligned} P_2^{(1)} &= (6.809378186)(1.01)^2 \\ &\quad + (1.01)(1.03)\left[-4.587155963\cos\theta_{21} \right. \\ &\qquad \left. + 15.290519878\sin\theta_{21}\right] \\ &\quad + (1.01)v_3\left[-2.222222222\cos\theta_{23} \right. \\ &\qquad \left. + 6.666666667\sin\theta_{23}\right] \\ &= 0.540176988. \end{aligned}$$

The trigonometric terms now use the updated angles. The linear prediction is not substituted here.

At state 1: Recalculate P3 and Q3

$$\theta_{31} = -0.060900164, \qquad \theta_{32} = -0.067005338, \qquad v_3 = 0.967950421.$$

$$\begin{aligned} P_3^{(1)} &= 3.650793651 v_3^2 \\ &\quad + 1.03 v_3(-1.428571429 \cos\theta_{31} + 4.285714286 \sin\theta_{31}) \\ &\quad + 1.01 v_3(-2.222222222 \cos\theta_{32} + 6.666666667 \sin\theta_{32}) \\ &= -0.865171243, \end{aligned}$$

$$\begin{aligned} Q_3^{(1)} &= 10.952380952 v_3^2 \\ &\quad + 1.03 v_3(-1.428571429 \sin\theta_{31} - 4.285714286 \cos\theta_{31}) \\ &\quad + 1.01 v_3(-2.222222222 \sin\theta_{32} - 6.666666667 \cos\theta_{32}) \\ &= -0.274057817. \end{aligned}$$

At state 1: Test the original residual

Each residual is the network term plus demand minus generation.

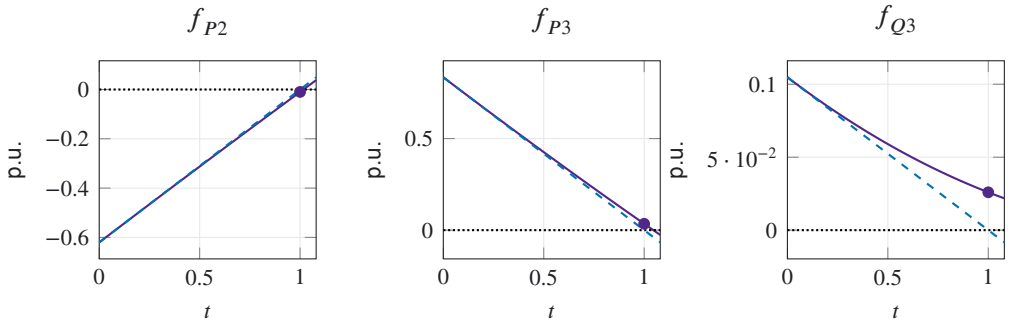
Row	Network term	Fixed term	Residual
f_{P2}	0.540176988	−0.55	−0.009823012
f_{P3}	-0.865171243	+0.90	0.034828757
f_{Q3}	-0.274057817	+0.30	0.025942183

$$\|\mathbf{f}(\mathbf{x}^{(1)})\|_\infty = 3.482876 \times 10^{-2} > 10^{-3} \text{ p.u.}$$

The residual remains above tolerance. Rebuild the Jacobian at this updated state.

Why the first correction leaves a residual

Along the first correction direction, $\mathbf{x}(t) = \mathbf{x}^{(0)} + t\Delta\mathbf{x}^{(0)}$. The new state is at $t = 1$.



Solid: original residuals. Dashed: linear predictions. Dotted: zero residual.

At t=1 the linear model reaches zero. Curvature leaves a residual in the original equations.

Iteration 1: Inputs for the new Jacobian

At state 1, use $v_2 = 1.01$, $v_3 = 0.967950421$, and angles in radians.

$$\theta_2 - \theta_3 = 0.006105173 - (-0.060900164) = +0.067005338,$$
$$\theta_3 - \theta_2 = -0.060900164 - 0.006105173 = -0.067005338.$$

For H_{22}, also evaluate Q_2 at these same voltages:

$$\begin{aligned} Q_2 &= -B_{22}v_2^2 + v_2v_1[G_{21}\sin(\theta_2 - \theta_1) - B_{21}\cos(\theta_2 - \theta_1)] \\ &\quad + v_2v_3[G_{23}\sin(\theta_2 - \theta_3) - B_{23}\cos(\theta_2 - \theta_3)] \\ &= 21.9571865(1.01)^2 \\ &\quad + (1.01)(1.03)[-4.5871560\sin(0.0061052) - 15.2905199\cos(0.0061052)] \\ &\quad + (1.01)(0.9679504)[-2.2222222\sin(0.0670053) - 6.6666667\cos(0.0670053)] \\ &\approx -0.185407505. \end{aligned}$$

Iteration 1: The P2 derivative row

$$
\begin{aligned}
H_{22} &= -Q_2 - B_{22}v_2^2 \\
&= -(-0.185407505) - (-21.957186544)(1.01)^2 = 22.583933
\end{aligned}
$$

$$
\begin{aligned}
H_{23} &= v_2 v_3 \left[G_{23}\sin(\theta_2 - \theta_3) - B_{23}\cos(\theta_2 - \theta_3)\right] \\
&= (1.01)(0.967950421)\left[-2.222222222\sin(0.067005338)\right. \\
&\qquad \left. - 6.666666667\cos(0.067005338)\right] = -6.648368
\end{aligned}
$$

$$
\begin{aligned}
N_{23} &= v_2\left[G_{23}\cos(\theta_2 - \theta_3) + B_{23}\sin(\theta_2 - \theta_3)\right] \\
&= (1.01)\left[-2.222222222\cos(0.067005338)\right. \\
&\qquad \left. + 6.666666667\sin(0.067005338)\right] = -1.788576.
\end{aligned}
$$

Iteration 1: The P3 derivative row

$$
\begin{aligned}
H_{32} &= v_3 v_2 \left[G_{32} \sin(\theta_3 - \theta_2) - B_{32} \cos(\theta_3 - \theta_2) \right] \\
&= (0.967950421)(1.01) \left[-2.222222222 \sin(-0.067005338) \right. \\
&\qquad \left. - 6.666666667 \cos(-0.067005338) \right] = -6.357446
\end{aligned}
$$

$$
\begin{aligned}
H_{33} &= -Q_3 - B_{33} v_3^2 \\
&= -(-0.274057817) - (-10.952380952)(0.967950421)^2 = 10.535650
\end{aligned}
$$

$$
\begin{aligned}
N_{33} &= \frac{P_3}{v_3} + G_{33} v_3 \\
&= \frac{-0.865171243}{0.967950421} + (3.650793651)(0.967950421) = 2.639970.
\end{aligned}
$$

Iteration 1: The Q3 derivative row

$$
\begin{aligned}
M_{32} &= -v_3 v_2 \left[G_{32} \cos(\theta_3 - \theta_2) + B_{32} \sin(\theta_3 - \theta_2) \right] \\
&= -(0.967950421)(1.01) \left[-2.222222222 \cos(-0.067005338) \right. \\
&\qquad \left. + 6.666666667 \sin(-0.067005338) \right] = 2.604019
\end{aligned}
$$

$$
\begin{aligned}
M_{33} &= P_3 - G_{33} v_3^2 \\
&= -0.865171243 - (3.650793651)(0.967950421)^2 = -4.285702
\end{aligned}
$$

$$
\begin{aligned}
L_{33} &= \frac{Q_3}{v_3} - B_{33} v_3 \\
&= \frac{-0.274057817}{0.967950421} - (-10.952380952)(0.967950421) = 10.318230.
\end{aligned}
$$

Iteration 1: Rebuild and solve

Assemble the coefficients just evaluated at

$$\mathbf{x}^{(1)} = \begin{bmatrix} 0.006105173 & -0.060900164 & 0.967950421 \end{bmatrix}^{\mathsf{T}}.$$

$$\begin{bmatrix} 22.583933 & -6.648368 & -1.788576 \\ -6.357446 & 10.535650 & 2.639970 \\ 2.604019 & -4.285702 & 10.318230 \end{bmatrix} \Delta\mathbf{x}^{(1)} = -\underbrace{\begin{bmatrix} -0.009823012 \\ 0.034828757 \\ 0.025942183 \end{bmatrix}}_{\mathbf{f}(\mathbf{x}^{(1)})}.$$

$$\Delta\mathbf{x}^{(1)} = -(\mathbf{J}^{(1)})^{-1}\mathbf{f}(\mathbf{x}^{(1)}) = \begin{bmatrix} -0.000677730 \\ -0.002832790 \\ -0.003519776 \end{bmatrix}.$$

Use the same state for both the Jacobian and the residual.

Correction 2: Add the new state change

$$\mathbf{x}^{(2)} = \mathbf{x}^{(1)} + \Delta\mathbf{x}^{(1)} = \begin{bmatrix} 0.006105173 \\ -0.060900164 \\ 0.967950421 \end{bmatrix} + \begin{bmatrix} -0.000677730 \\ -0.002832790 \\ -0.003519776 \end{bmatrix} = \begin{bmatrix} 0.005427443 \\ -0.063732955 \\ 0.964430645 \end{bmatrix}.$$

$$\theta_{21} = 0.005427443, \quad \theta_{31} = -0.063732955, \quad \theta_{32} = -0.069160398.$$

The first two entries are radians. The third entry is the PQ voltage magnitude in per unit.

At state 2: Evaluate the power terms

Each row adds the self term and two neighbor terms at state 2.

Row	Self	Neighbor 1	Other neighbor	Sum
P_2	6.946246687	-4.685615622	-1.710678095	0.549952969
P_3	3.395699805	-1.687354589	-2.608194353	-0.899849137
Q_3	10.187099414	-4.158247424	-6.328722630	-0.299870640

$$P_i = G_{ii}v_i^2 + \sum_{k\neq i} v_i v_k [G_{ik}\cos(\theta_i - \theta_k) + B_{ik}\sin(\theta_i - \theta_k)],$$

$$Q_i = -B_{ii}v_i^2 + \sum_{k\neq i} v_i v_k [G_{ik}\sin(\theta_i - \theta_k) - B_{ik}\cos(\theta_i - \theta_k)].$$

For P_2, the other neighbor is bus 3. For P_3, Q_3, it is bus 2.

At state 2: Test the original residual

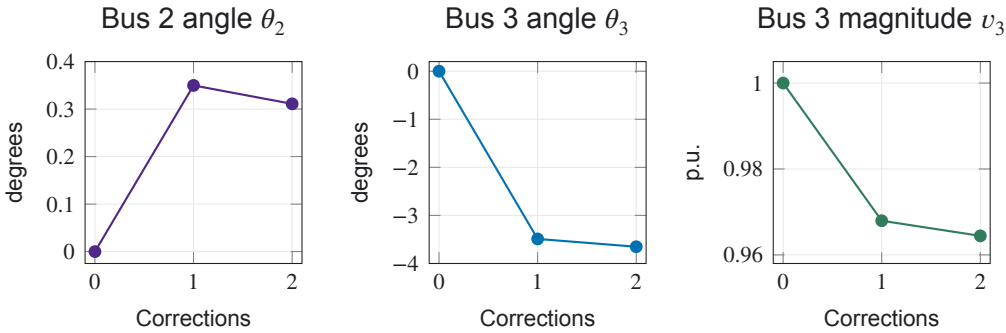
Each residual is the network term plus demand minus generation.

Row	Network term	Fixed term	Residual
f_{P2}	0.549952969	−0.55	−0.000047031
f_{P3}	-0.899849137	+0.90	0.000150863
f_{Q3}	-0.299870640	+0.30	0.000129360

$$\|\mathbf{f}(\mathbf{x}^{(2)})\|_\infty = 1.508627 \times 10^{-4} \leq 10^{-3} \text{ p.u.}$$

Accept after two corrections at this example's tolerance. A stricter tolerance may require another correction.

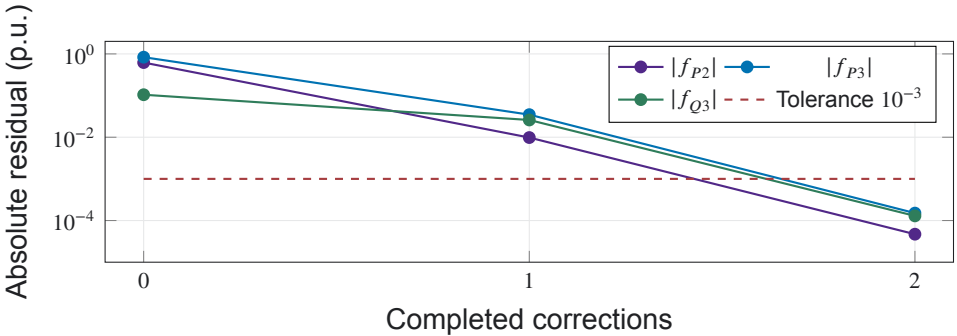
Voltage changes after correction 2



$$\mathbf{x}^{(2)} = \begin{bmatrix} 0.005427443 & -0.063732955 & 0.964430645 \end{bmatrix}^{\mathsf{T}}.$$

The plots show degrees for angles. The state vector below uses radians and per unit.

Convergence of every selected equation



$$\|\mathbf{f}\|_{\infty}: \quad 0.834921, \quad 0.0348288, \quad 1.50863 \times 10^{-4}.$$

All three residuals must pass. The first acceptable state is after correction 2.

PV capability at every tested state

Bus 2 has no local load, so $Q_{G2} = Q_2$. The allowed range is $[-0.25, +0.50]$ p.u.

State	Q_2 (p.u.)	Q_{G2} (Mvar)	Check
0	-0.241535168	-24.153517	Within limits
1	-0.185407505	-18.540750	Within limits
2	-0.161762000	-16.176200	Within limits

The PV model stays active. No Q2 row or v2 correction column is added in this example.

Accepted state: Calculate the slack power

Use the accepted state $\mathbf{x}^{(2)}$ to calculate the slack output:

$$I_1 = Y_{11}V_1 + Y_{12}V_2 + Y_{13}V_3 = 0.367708642 - \text{j}0.532837202.$$

$$S_1 = V_1 I_1^* = 1.03\left(0.367708642 + \text{j}0.532837202\right) = 0.378739902 + \text{j}0.548822318.$$

$$P_{G1} = 37.873990 \text{ MW}, \qquad Q_{G1} = 54.882232 \text{ Mvar}.$$

The slack voltage remains fixed. These outputs use the voltage state accepted at the stated tolerance.

Example: Accepted voltages and outputs

Blue: fixed voltages Purple: values evaluated at $\mathbf{x}^{(2)}$

Bus	v (p.u.)	Angle (deg.)	Net P (MW)	Net Q (Mvar)
1	1.030000	0.000000	37.87399	54.88223
2	1.010000	0.310970	54.99530	-16.17620
3	0.964431	-3.651629	-89.98491	-29.98706

$$\|\mathbf{f}(\mathbf{x}^{(2)})\|_\infty = 1.5086 \times 10^{-4} < 10^{-3} \text{ p.u.}.$$

The displayed powers come from the accepted voltages. Their remaining differences from the scheduled injections are within the stated tolerance.

Bus 2 Q is within $[-25, 50]$ Mvar. The PV model remains consistent with this accepted state.

Summary

1. Evaluate $\mathbf{f} = [P_2 - 0.55,\ P_3 + 0.90,\ Q_3 + 0.30]^{\mathsf{T}}$.
2. Evaluate each Jacobian coefficient at those same voltages.
3. Solve $\mathbf{J}\Delta\mathbf{x} = -\mathbf{f}$ and add the correction to the voltage state.
4. Recalculate the original nonlinear equations after every update.
5. Accept only after all selected residuals and active controls pass.

Two corrections meet this example's tolerance. The path matters: each new state determines the next residual and Jacobian.